Math 4650

Topic 2 – Sequences

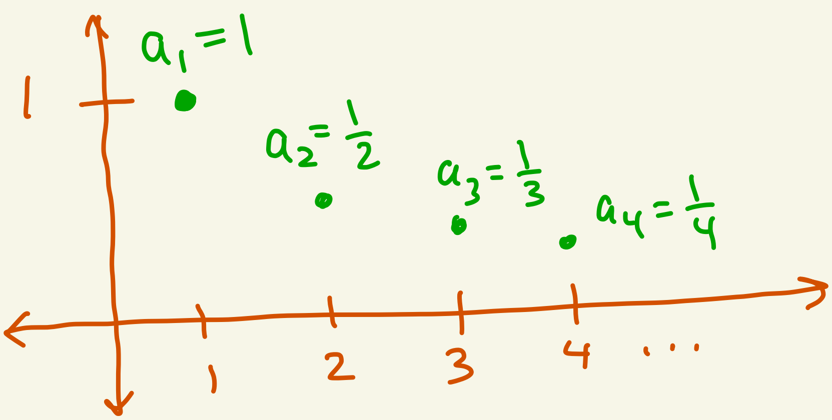
<u>Def:</u> A sequence of real numbers written (a_n) or $(a_n)_{n=1}^{\infty}$ is an ordered list of real numbers

$$a_1, a_2, a_3, a_4, a_5, \ldots$$

Note: The sequence can start from an index that isn't 1, for example you can have $(a_n)_{n=2}^{\infty}$ giving $a_2, a_3, a_4, a_5, \ldots$

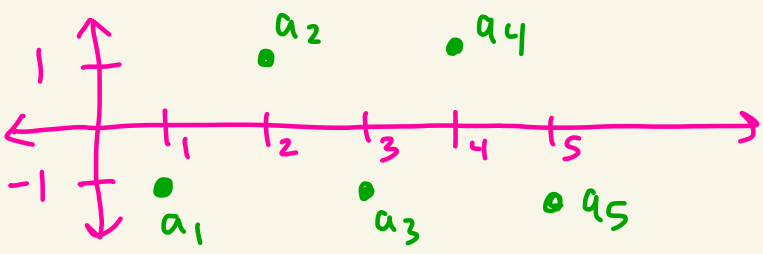
<u>Ex:</u> $a_n = \frac{1}{n}$

sequence: $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \ldots$



<u>Ex:</u> $a_n = (-1)^n$

sequence: $-1, 1, -1, 1, -1, 1, \ldots$

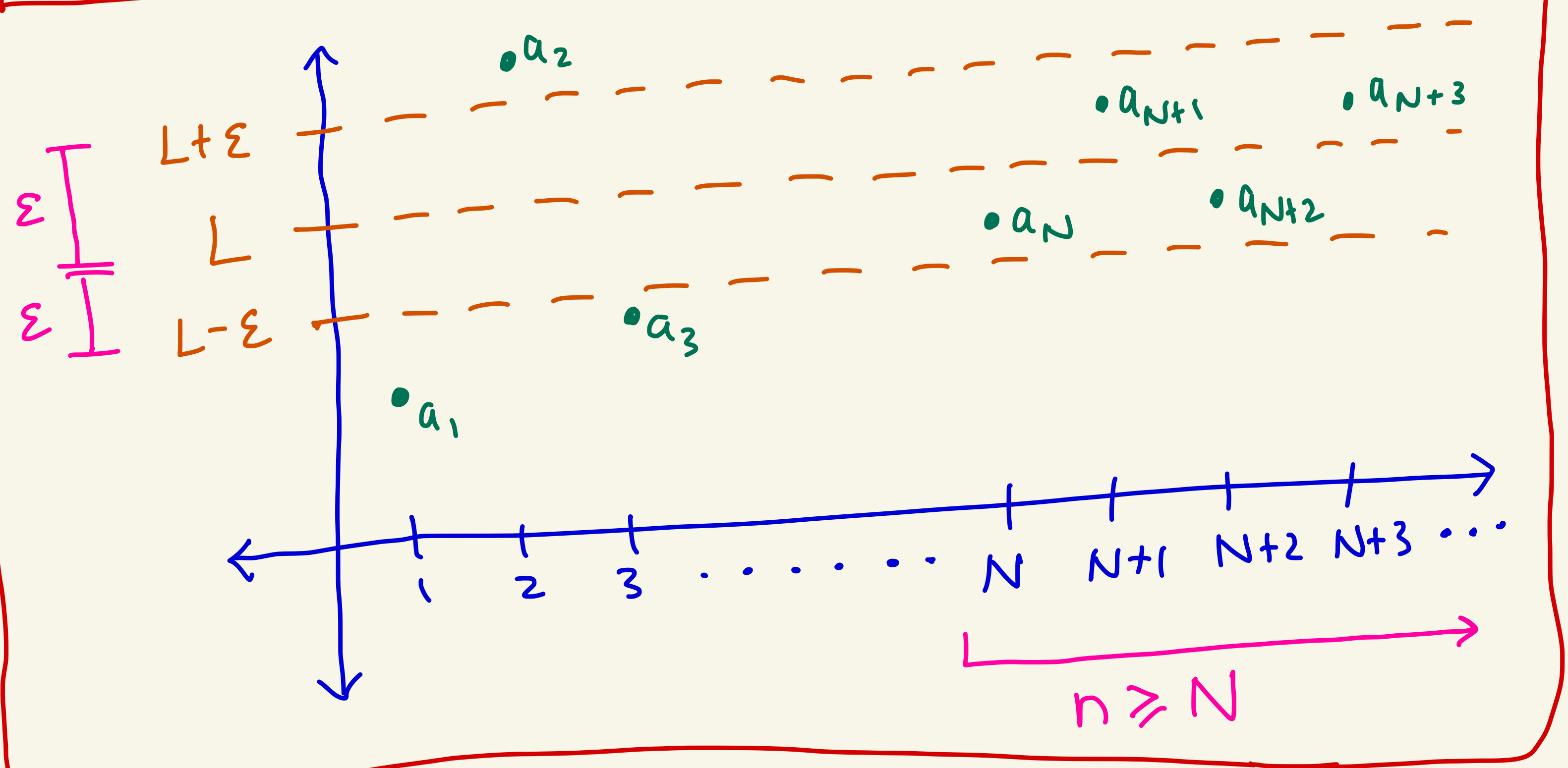


Def: A sequence of real numbers (a_n) is said to converge to a limit $L \in \mathbb{R}$ if for every real number $\varepsilon > 0$, there exists a natural number N such that if $n \geq N$, then $|a_n - L| < \varepsilon$

If this is the case, we write $\lim_{n\to\infty} a_n = L$ or $a_n \to L$.

If no such L exists, then we say that (a_n) diverges

PICTURE

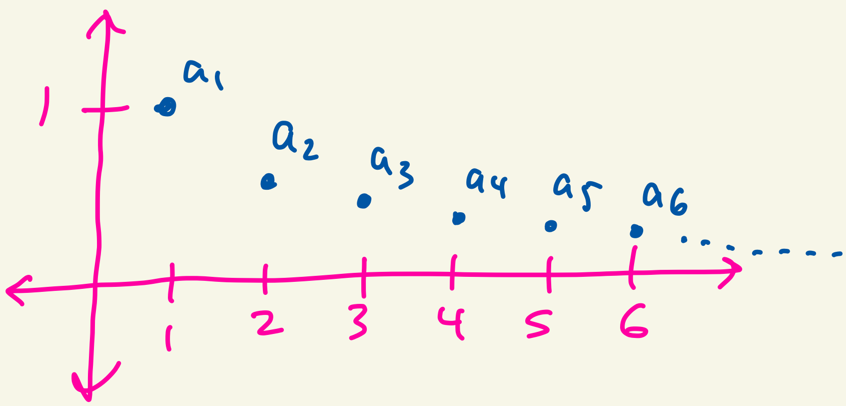


Note: N depends on ε. You get a different N for each ε. Some people write $N(\varepsilon)$ instead of N, but we won't do that.

Ex: Consider $a_n = \frac{1}{n}$.

sequence: $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \frac{1}{8}, \frac{1}{9}, \ldots$

It seems that the limit is $L = 0$.

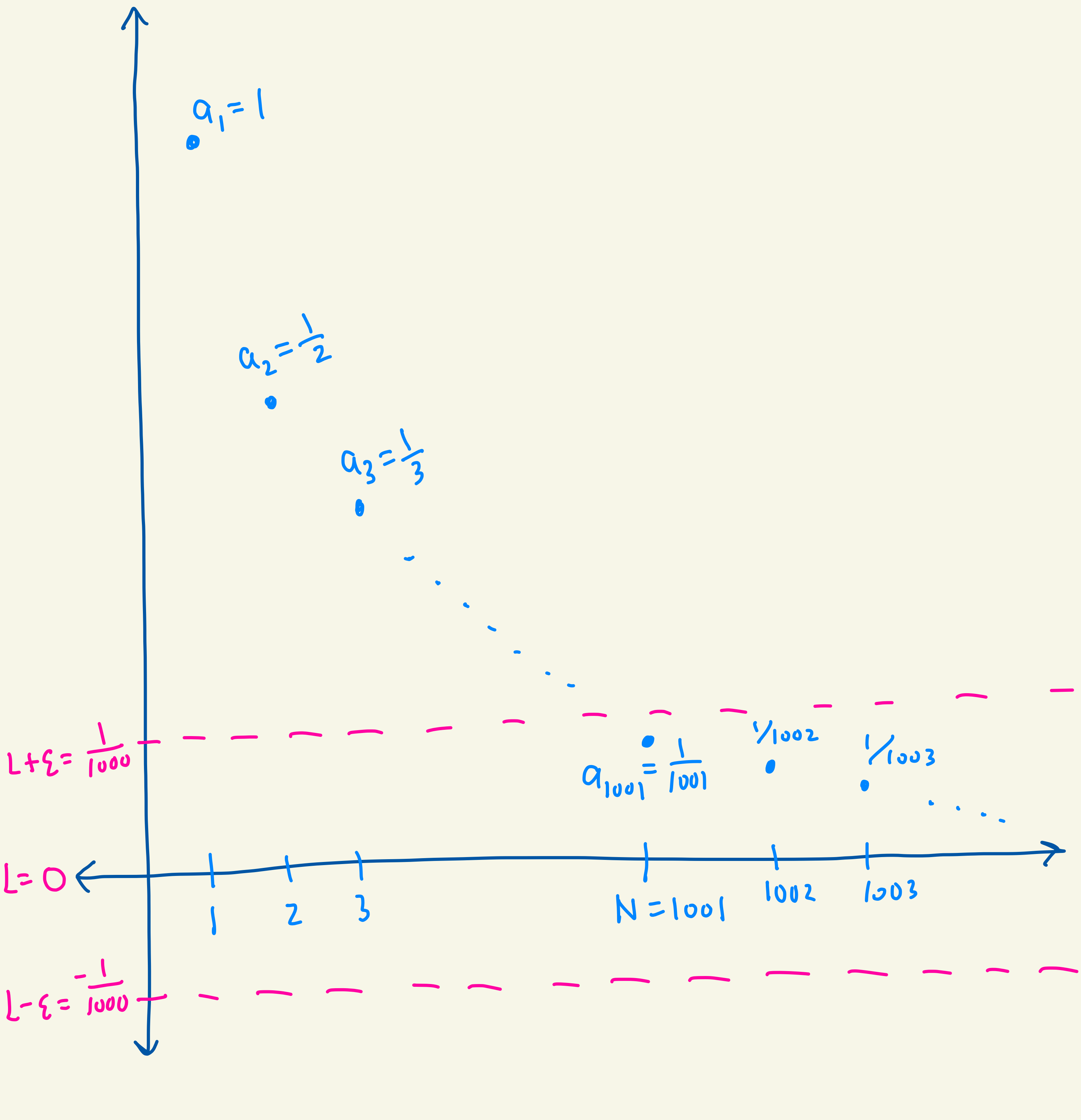


Before we show that $\lim_{n \to \infty} \frac{1}{n} = 0$, let's get a feel for the definition. With $L = 0$ we need to show: "for every $\varepsilon > 0$, there exists $N > 0$ where if $n \geq N$, then $\left| \frac{1}{n} - 0 \right| < \varepsilon$"

Let's say $\varepsilon = \frac{1}{1000} = 0.001$

Then if $N = 1001$ we have that

if $n \geq \underbrace{1001}_{N}$, then $\left|\frac{1}{n} - 0\right| = \left|\frac{1}{n}\right| = \frac{1}{n} \leq \frac{1}{1001} < \varepsilon$



<u>Claim</u>: $\lim_{n\to\infty} \frac{1}{n} = 0$

<u>proof:</u>

Let $\varepsilon > 0$.

Pick a natural number N where $N > \frac{1}{\varepsilon}$.

Then if $n \geq N$ we have that

$$\underbrace{\left|\frac{1}{n} - 0\right|}_{|a_n - L|} = \left|\frac{1}{n}\right| = \frac{1}{n} \leq \frac{1}{N} < \varepsilon.$$

$$\boxed{\frac{1}{n} > 0}$$

$\blacksquare$

<u>Ex:</u> If $c \in \mathbb{R}$ then $\lim_{n\to\infty} c = c$.

<u>proof:</u> Let $a_n = c$ for all n.

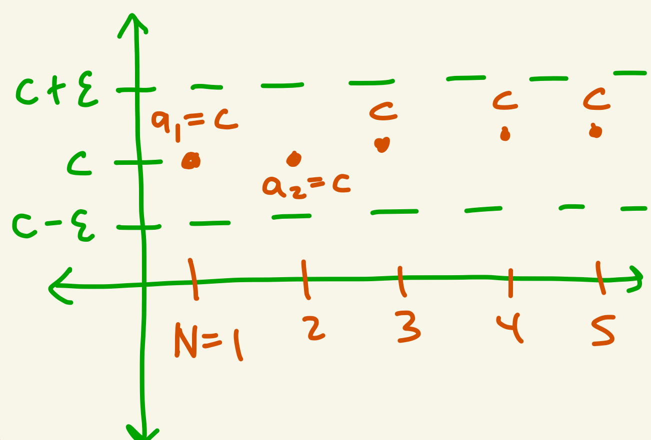
Let $\varepsilon > 0$.

Set $N = 1$.

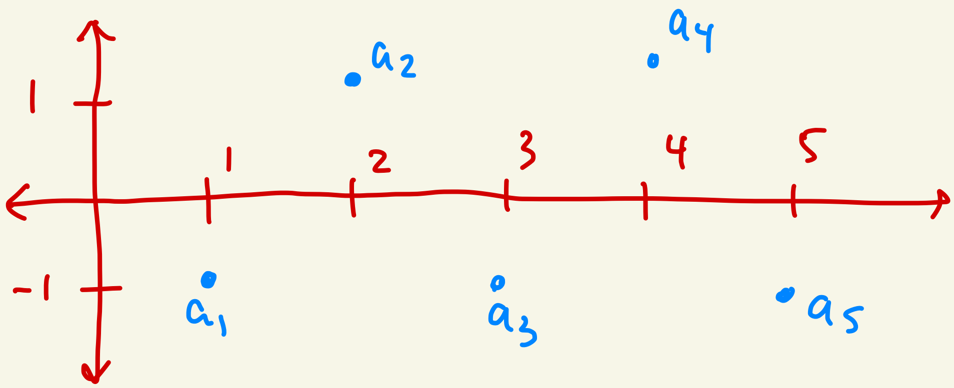
Then if $n \geq 1$ we have

$$|a_n - c| = |c - c| = 0 < \varepsilon.$$

$\blacksquare$



Ex: Consider $a_n = (-1)^n$



Let's show that this sequence diverges, that is, it has no limit L.

Claim: $a_n = (-1)^n$ diverges.

proof: We prove this by contradiction. Suppose (a_n) converges to some $L \in \mathbb{R}$.

Let $\varepsilon = 1$.

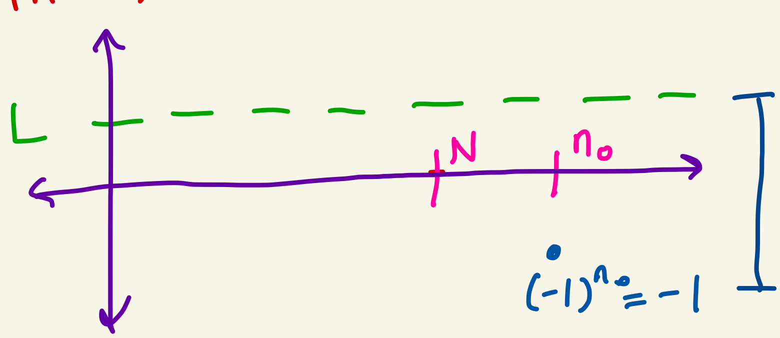
Then since $(-1)^n \to L$ there must exist N where if $n \geq N$ then $|(-1)^n - L| < 1$.

case 1: Suppose $L \geq 0$.

Pick an odd integer n_0 with $n_0 \geq N$.

Then, $|(-1)^{n_0} - L| = |-1-L| = -(-1-L) = 1+L \geq 1 = \varepsilon$

$-1-L<0$ since $L \geq 0$



this distance is $|(-1)^{n_0} - L| \geq \varepsilon$ with $n_0 \geq N$

We get a contradiction.

<u>Case 2:</u> Suppose $L < 0$.

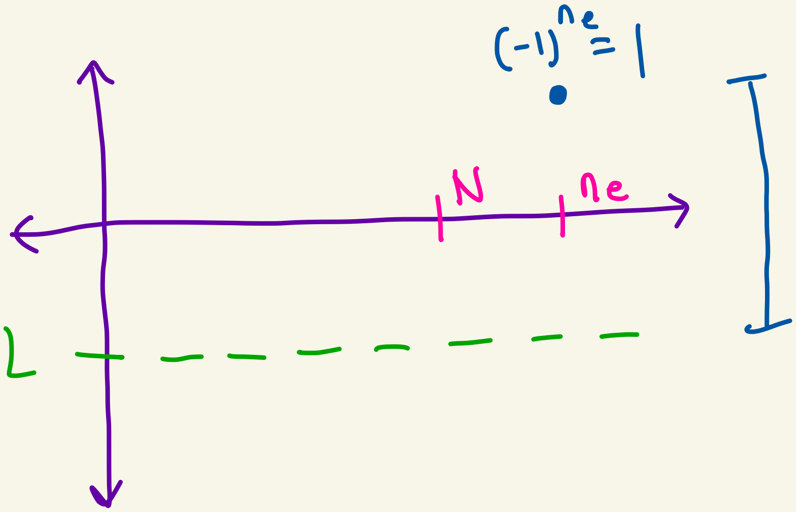
Pick an even integer n_e with $n_e \geq N$.

Then

$$|(-1)^{n_e} - L| = |1 - L| = 1 - L \geq 1 = \varepsilon$$

$1 - L > 0$ since $L < 0$

because $-L > 0$



this distance is $|(-1)^{n_e} - L| \geq \varepsilon$ with $n_e \geq N$

Again a contradiction

In either case we get a contradiction.

Thus, $(-1)^n$ diverges. $\blacksquare$

Ex: $\lim_{n\to\infty} \frac{n}{n+1} = 1.$

proof:

Let $\varepsilon > 0$.

Note that

$$\underbrace{\left|\frac{n}{n+1} - 1\right|}_{|a_n - L|} = \left|\frac{n}{n+1} - \frac{n+1}{n+1}\right| = \left|\frac{-1}{n+1}\right| = \frac{1}{n+1} < \frac{1}{n}$$

side commentary:
We need N where if $n \geq N$ then $\frac{1}{n} < \varepsilon$.
So, need $\frac{1}{\varepsilon} < n$.
Pick $N > \frac{1}{\varepsilon}$

Thus, $\left|\frac{n}{n+1} - 1\right| < \frac{1}{n}$.

Pick N so that $N > \frac{1}{\varepsilon}$.

Then if $n \geq N$ we get that

$$\left|\frac{n}{n+1} - 1\right| \underset{\text{from above}}{<} \frac{1}{n} \underset{\text{since } n \geq N}{\leq} \frac{1}{N} \underset{\text{since } N > \frac{1}{\varepsilon}}{<} \varepsilon$$

$\blacksquare$

<u>Theorem:</u> Limits of sequences are unique.

That is, if $\lim_{n\to\infty} a_n = L_1$ and $\lim_{n\to\infty} a_n = L_2$

then $L_1 = L_2$.

<u>proof:</u>

Let $\varepsilon > 0$.

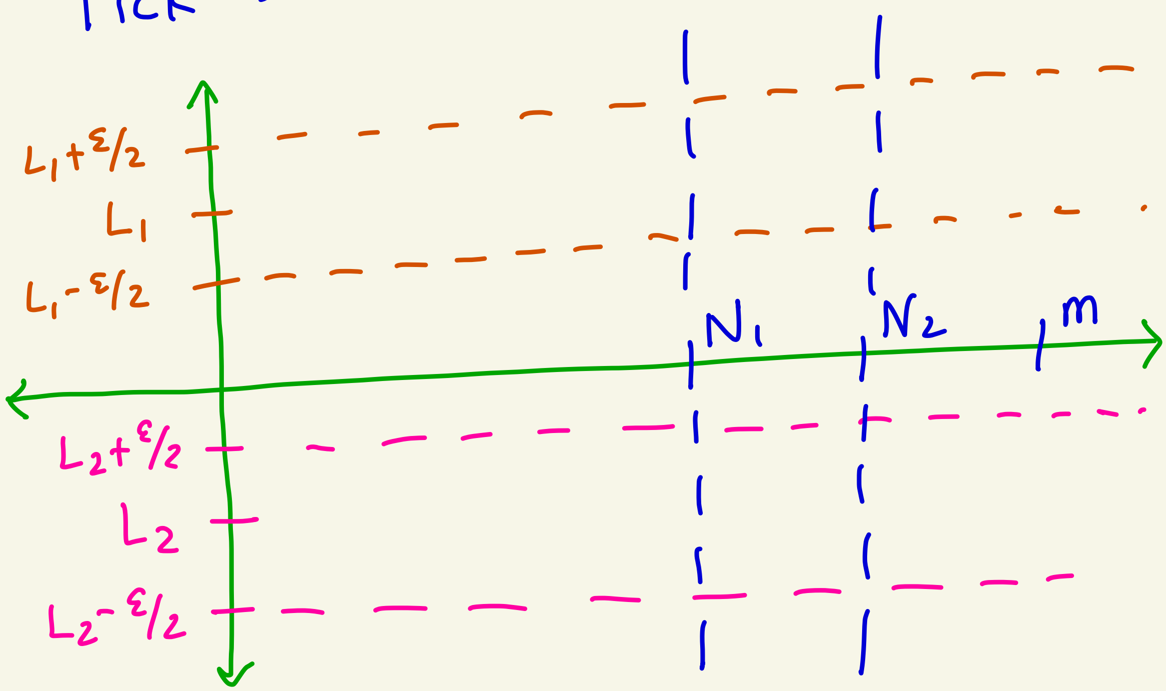
Since $\lim_{n\to\infty} a_n = L_1$ there exists N_1 where

if $n \geq N_1$ then $|a_n - L_1| < \varepsilon/2$.

Since $\lim_{n\to\infty} a_n = L_2$ there exists N_2 where

if $n \geq N_2$ then $|a_n - L_2| < \varepsilon/2$.

Pick some m with $m \geq N_1$ and $m \geq N_2$.



This picture can't actually happen since L_1 will equal L_2 but it gives an idea of the construction

Then,

$$
\begin{aligned}
|L_1 - L_2| &= |L_1 - a_m + a_m - L_2| \\
&\underset{\Delta\text{-inequality}}{\leq} |L_1 - a_m| + |a_m - L_2| \\
&\underset{|-x| = |x|}{=} |a_m - L_1| + |a_m - L_2| \\
&\underset{m \geq N_1 \ \& \ m \geq N_2}{<} \varepsilon/2 + \varepsilon/2 = \varepsilon
\end{aligned}
$$

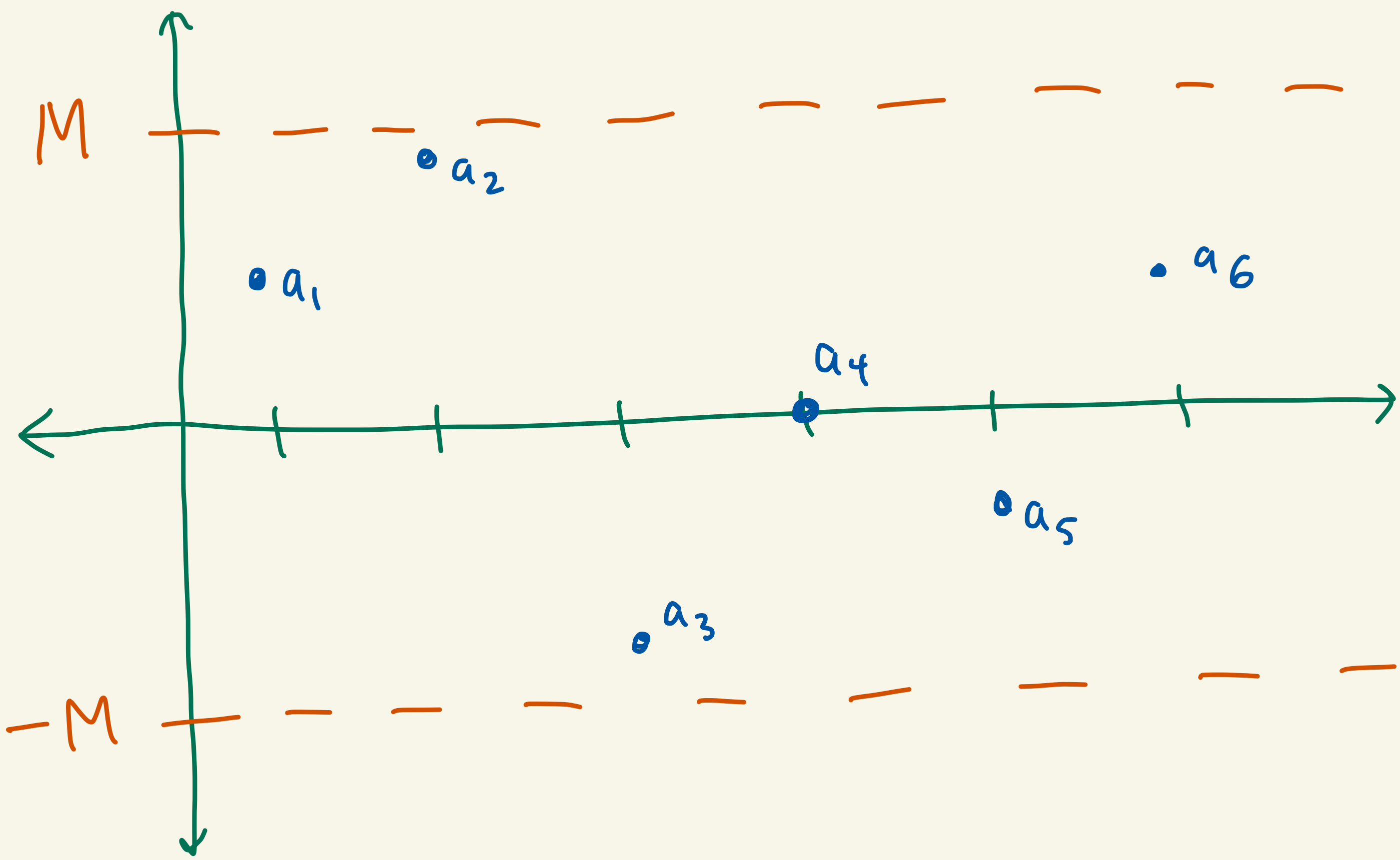
We have shown that for any $\varepsilon > 0$ we have that $|L_1 - L_2| < \varepsilon$.

Thus, $|L_1 - L_2| = 0$.

So, $L_1 - L_2 = 0$.

Thus, $L_1 = L_2$. $\blacksquare$

Def: A sequence (a_n) of real numbers is **bounded** if there exists a real number $M > 0$ where $\underbrace{|a_n| \le M}_{\text{same as } -M \le a_n \le M}$ for all n.



Theorem: If (a_n) converges, then (a_n) is bounded.

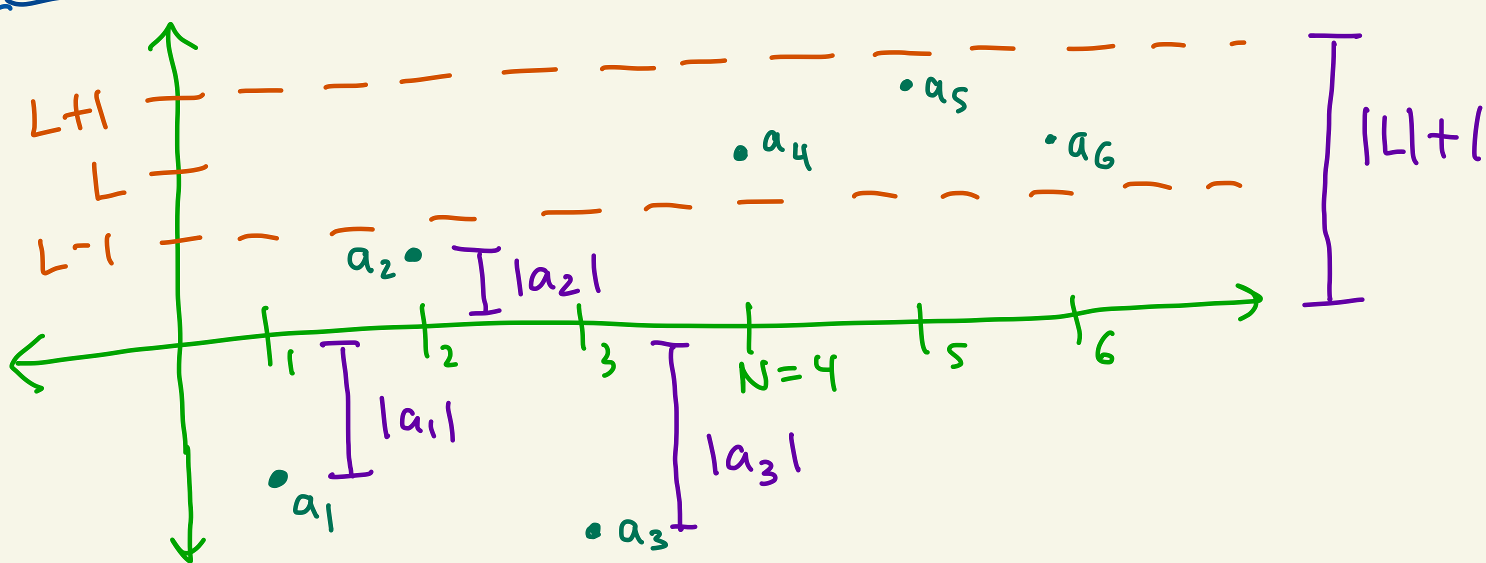
proof:
Suppose (a_n) converges and $\lim_{n\to\infty} a_n = L$.

Pick $\varepsilon = 1$.
Then there exists N where
if $n \geq N$, then $|a_n - L| < 1$.

So if $n \geq N$, then
$$|a_n| = |a_n - L + L| \leq |a_n - L| + |L| < 1 + |L|$$

Let
$$M = \max\{|a_1|, |a_2|, \ldots, |a_{N-1}|, 1+|L|\}$$

Picture with $N=4$



Then, $|a_n| \leq M$ for all n.

So, (a_n) is bounded. $\blacksquare$

<u>Corollary:</u> If (a_n) is not bounded, then (a_n) diverges. $\blacksquare$

<u>proof:</u> Contrapositive of above theorem. $\blacksquare$

<u>Ex:</u> (n^2) diverges.

<u>proof:</u> We show the sequence $a_n = n^2$ is unbounded.

Suppose $M > 0$.

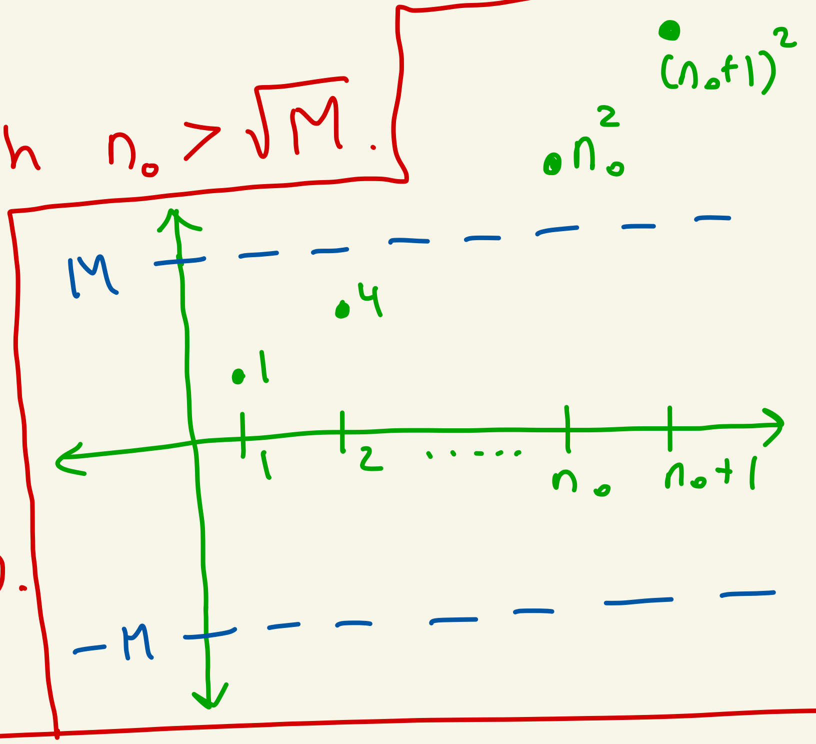
Pick an integer n_0 with $n_0 > \sqrt{M}$.

Then, $n_0^2 > M$.

So, $|n_0^2| > M$.

Thus, (n^2) cannot be bounded by any $M > 0$.

So, (n^2) diverges. $\blacksquare$



Algebra of sequences theorem:

Let (a_n) and (b_n) be convergent sequences with $\lim_{n\to\infty} a_n = A$ and $\lim_{n\to\infty} b_n = B$. Let $\alpha \in \mathbb{R}$.

Then:

① (αa_n) converges and $\lim_{n\to\infty} \alpha a_n = \alpha A$

② $(a_n + b_n)$ converges and $\lim_{n\to\infty} (a_n + b_n) = A + B$

③ $(a_n b_n)$ converges and $\lim_{n\to\infty} a_n b_n = AB$

④ If $B \neq 0$ and $b_n \neq 0$ for all n, then $\left(\frac{1}{b_n}\right)$ converges with $\lim_{n\to\infty} \frac{1}{b_n} = \frac{1}{B}$

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proof:

① If $\alpha = 0$ then $\lim_{n\to\infty} \alpha a_n = \lim_{n\to\infty} 0 = 0$.

So assume $\alpha \neq 0$.

Let $\varepsilon > 0$.

Since $a_n \to A$ there exists N where if $n \geq N$, then $|a_n - A| < \frac{\varepsilon}{|\alpha|}$.

If $n \geq N$ then

$$|\alpha a_n - \alpha A| = |\alpha||a_n - A|$$
$$< |\alpha| \cdot \frac{\varepsilon}{|\alpha|}$$
$$= \varepsilon$$

So if $n \geq N$, then $|\alpha a_n - \alpha A| < \varepsilon$.

Thus, $\alpha a_n \to \alpha A$.

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② Let $\varepsilon > 0$.

Since $a_n \to A$ there exists N_1 where if $n \geq N_1$, then $|a_n - A| < \varepsilon/2$.

Since $b_n \to B$, there exists N_2 where if $n \geq N_2$, then $|b_n - B| < \varepsilon/2$.

Let $N = \max\{N_1, N_2\}$.

If $n \geq N$, then

$$|(a_n + b_n) - (A + B)| = |(a_n - A) + (b_n - B)|$$
$$\leq |a_n - A| + |b_n - B|$$
$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$$
$$= \varepsilon$$

Thus, $a_n + b_n \to A + B$.

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3 Let $\varepsilon > 0$.

Note that

$$|a_n b_n - AB| = |a_n b_n - b_n A + b_n A - AB|$$
$$\leq |a_n b_n - b_n A| + |b_n A - AB|$$
$$= |b_n|\,|a_n - A| + |A|\,|b_n - B|$$

Since (b_n) converges it is bounded so there exists $M > 0$ where $|b_n| \leq M$ for all n.

Since $a_n \to A$ there exists N_1 where if $n \geq N_1$ then $|a_n - A| < \frac{\varepsilon}{2M}$.

Since $b_n \to B$ there exists N_2 where if $n \geq N_2$ then $|b_n - B| < \frac{\varepsilon}{2\underbrace{(|A|+1)}}$

$|A|+1$ is used in case $A=0$

Let $N = \max\{N_1, N_2\}$

If $n \geq N$, then

$$|a_n b_n - AB| \leq |b_n|\,|a_n - A| + |A|\,|b_n - B|$$

$$< M \cdot \frac{\varepsilon}{2M} + |A| \cdot \frac{\varepsilon}{2(|A|+1)}$$

$$= \frac{\varepsilon}{2} + \frac{\varepsilon}{2}\left(\frac{|A|}{|A|+1}\right)$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$$

$$= \varepsilon.$$

So if $n \geq N$ then $|a_n b_n - AB| < \varepsilon$.

Thus, $a_n b_n \to AB$.

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(4) Let $\varepsilon > 0$.

Note that

$$\left|\frac{1}{b_n} - \frac{1}{B}\right| = \left|\frac{B - b_n}{b_n B}\right|$$

$$= \frac{|B - b_n|}{|b_n B|}$$

$$= \frac{|b_n - B|}{|b_n||B|}$$

$b_n \neq 0$ for all n so this is all defined

Since $b_n \to B$ we can find N_1 where if

$n \geq N_1$ then $|b_n - B| < \frac{|B|}{2}$.

Thus, if $n \geq N_1$ then

$$\big|\,|b_n| - |B|\,\big| \underset{\substack{\uparrow \\ \boxed{\text{HW 1}}}}{\leq} |b_n - B| < \frac{|B|}{2}$$

$|x| < c$
iff
$-c < x < c$

implying

$$-\frac{|B|}{2} < |b_n| - |B| < \frac{|B|}{2}$$

So if $n \geq N_1$ then

$$\underbrace{\frac{|B|}{2} < |b_n|}_{\text{we will use this side below}} < \frac{3|B|}{2}$$

Again since $b_n \to B$ we can find N_2 where if $n \geq N_2$ then $|b_n - B| < \frac{\varepsilon}{2}|B|^2$

Let $N = \max\{N_1, N_2\}$.

Then if $n \geq N$ we have

$$\left|\frac{1}{b_n} - \frac{1}{B}\right| = \frac{|b_n - B|}{|b_n||B|}$$

$$= \frac{1}{|b_n|} \frac{1}{|B|} |b_n - B|$$

$$\boxed{\begin{array}{l} |b_n| > \frac{|B|}{2} \\ \frac{1}{|b_n|} < \frac{2}{|B|} \end{array}}$$

$$< \frac{2}{|B|} \cdot \frac{1}{|B|} \cdot \frac{\varepsilon}{2} |B|^2$$

$$= \varepsilon$$

Thus if $n \geq N$, then $\left| \frac{1}{b_n} - \frac{1}{B} \right| < \varepsilon$.

So, $\frac{1}{b_n} \to \frac{1}{B}$.

$\blacksquare$

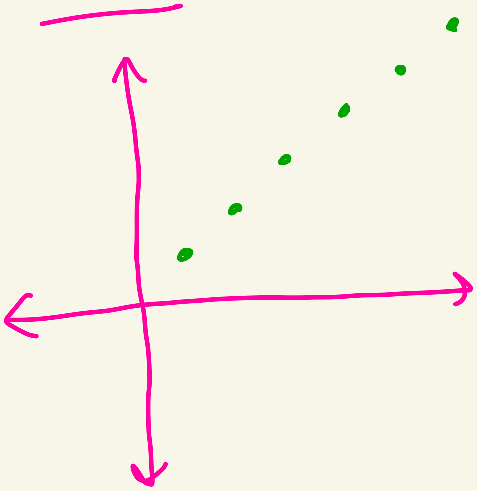
Def: Let (a_n) be a sequence of real numbers.

We say that (a_n) is monotone increasing if $a_n \leq a_{n+1}$ for all n.

We say that (a_n) is monotone decreasing if $a_{n+1} \leq a_n$ for all n.

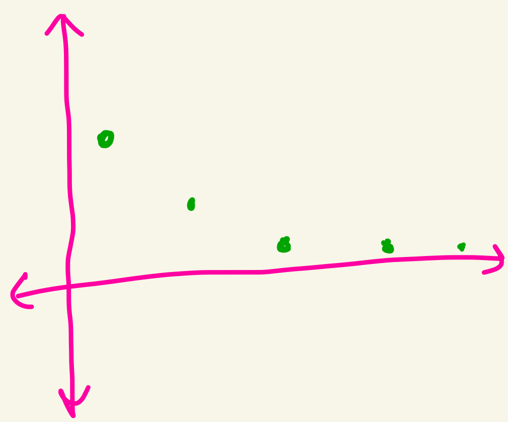
We say that (a_n) is monotone if it is either monotone increasing or monotone decreasing

Ex: $a_n = n$



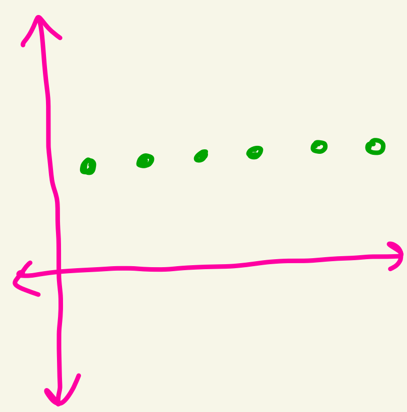
monotone increasing so its monotone

Ex: $a_n = \frac{1}{n}$



monotone decreasing so its monotone

Ex: $a_n = c$



monotone increasing and monotone decreasing

so its monotone

Monotone convergence theorem

If (a_n) is a bounded monotone sequence, then (a_n) converges.

<u>proof:</u> We will prove this for the case where (a_n) is monotone increasing. The monotone decreasing case is similar.

Since (a_n) is monotone increasing we know that $a_n \leq a_{n+1}$ for all n.

Since (a_n) is bounded there exists $M > 0$ where $|a_n| \leq M$ for all n.

Let $S = \{a_n \mid n \geq 1\} = \{a_1, a_2, a_3, \ldots\}$

Let $L = \sup(S)$

If (a_n) was monotone decreasing you'd set $L = \inf(S)$

We know that L exists by the completeness axiom because the set S is bounded above by M.

We will show $\lim\limits_{n \to \infty} a_n = L$.

Let $\varepsilon > 0$.

By the inf/sup theorem there exists N where $L-\varepsilon < a_N \leq L$.

Since a_n is monotone increasing we know $a_N \leq a_n$ for all $n \geq N$.

So if $n \geq N$, then

$$L-\varepsilon < a_N \leq a_n \leq L < L+\varepsilon$$

$\boxed{L = \sup\{a_1, a_2, a_3, \dots\}}$

So if $n \geq N$, then $\underbrace{|a_n - L| < \varepsilon}_{\text{because } L-\varepsilon < a_n < L+\varepsilon}$.

Thus, $\lim\limits_{n\to\infty} a_n = L$

$\blacksquare$

Application to finding square roots

Theorem: Let $a > 0$ be a real number.

Define the sequence:

$a_1 =$ any positive real number

$$a_{n+1} = \frac{1}{2}\left(a_n + \frac{a}{a_n}\right) \text{ for } n \geq 1$$

Note: Formula comes from Newton's method

Then:

① $(a_n)_{n=1}^{\infty}$ converges

② $\lim_{n\to\infty} a_n = \sqrt{a}$

③ $|a_n - \sqrt{a}| \leq \dfrac{a_n^2 - a}{a_n}$ when $n \geq 2$ ← error bound

Proof:

① We need two claims first.

Claim (i): $a_n \geq \sqrt{a}$ for $n \geq 2$

Let $n \geq 1$.

By def we have $2a_{n+1} = a_n + \dfrac{a}{a_n}$.

So, $a_n^2 - 2a_n a_{n+1} + a = 0$.

Thus, $x^2 - 2a_{n+1}x + a = 0$ has a real root $(x = a_n)$.

So, the discriminant must be non-negative.

That is, $4a_{n+1}^2 - 4a \geq 0$.

So, $a_{n+1}^2 \geq a$.

Thus, $a_{n+1} \geq \sqrt{a}$ for $n \geq 1$.

Claim (ii): $a_n \geq a_{n+1}$ for $n \geq 1$

from claim 1:
$a_n^2 \geq a > 0$
So, $a_n > 0$
and $a_n^2 - a \geq 0$

Let $n \geq 1$.
Then,

$$a_n - a_{n+1} = a_n - \frac{1}{2}a_n - \frac{1}{2}\frac{a}{a_n} = \frac{1}{2}\left(\frac{a_n^2 - a}{a_n}\right) \geq 0$$

Thus, $a_n \geq a_{n+1}$

We have shown that
$a_2 \geq a_3 \geq a_4 \geq a_5 \geq \cdots\cdots \geq \sqrt{a} > 0$
By the monotone convergence theorem,
$(a_n)_{n=1}^{\infty}$ converges.

2 Let $L = \lim_{n \to \infty} a_n$.

We know $a_{n+1} = \frac{1}{2}\left(a_n + \frac{a}{a_n}\right)$ for $n \geq 1$.
Taking the limit of both sides gives

$$L = \frac{1}{2}\left(L + \frac{a}{L}\right)$$

So, $2L^2 = L^2 + a$

Thus, $L^2 = a$.

And since $L > 0$ we must have $L = \sqrt{a}$.

[We used $a_n > 0 \to L > 0$. This follows from HW problems]

③ Let $n \geq 2$.

Then, $a_n \geq \sqrt{a} \geq \frac{a}{a_n}$

above

$a_n \geq \sqrt{a} = \frac{a}{\sqrt{a}}$

So, $\sqrt{a} \geq \frac{a}{a_n}$

Thus,

$$0 \leq a_n - \sqrt{a} \leq a_n - \frac{a}{a_n} = \frac{a_n^2 - a}{a_n}$$

So, $|a_n - \sqrt{a}| \leq \frac{a_n^2 - a}{a_n}$

$\blacksquare$

Ex: Let's approximate $\sqrt{2}$. Here $a=2$.

Set $a_1 = 1 > 0$.

We have:

$a_{n+1} = \frac{1}{2}\left(a_n + \frac{2}{a_n}\right)$	Error bound $\frac{a_n^2 - a}{a_n}$
$a_2 = \frac{1}{2}\left(1 + \frac{1}{2}\right) = \frac{3}{2} = 1.5$	$\frac{a_2^2 - 2}{a_2} = \frac{1.5^2 - 2}{1.5} \approx 0.1666\ldots$
$a_3 = \frac{1}{2}\left(\frac{3}{2} + \frac{2}{(3/2)}\right) = \frac{17}{12}$ $\approx 1.416666\ldots$	$\frac{a_3^2 - 2}{a_3} = \frac{1}{204} \approx 0.00490196\ldots$
$a_4 = \frac{1}{2}\left(\frac{17}{12} + \frac{2}{17/12}\right) = \frac{577}{408}$ $\approx 1.414215686\ldots$	$\frac{a_4^2 - 2}{a_4} = \frac{1}{235{,}416}$ $\approx 0.0000042477996\ldots$
$a_5 = \frac{1}{2}\left(\frac{577}{408} + \frac{2}{577/408}\right)$ $= \frac{665857}{470832} \approx 1.41421356237\ldots$	$\frac{a_5^2 - 2}{a_5} = \frac{1}{313{,}506{,}783{,}024}$ $\approx 3.1897 \times 10^{-12}$

We get rapid convergence here.

Def: Let (a_n) be a sequence of real numbers. Let $n_1 < n_2 < n_3 < n_4 < \ldots$ be a strictly increasing sequence of natural numbers.
Then the sequence $(a_{n_k})_{k=1}^{\infty}$ given by

$$a_{n_1}, a_{n_2}, a_{n_3}, a_{n_4}, \ldots$$

is called a **subsequence** of (a_n)

Ex:

sequence: $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \frac{1}{8}, \frac{1}{9}, \frac{1}{10}, \ldots$

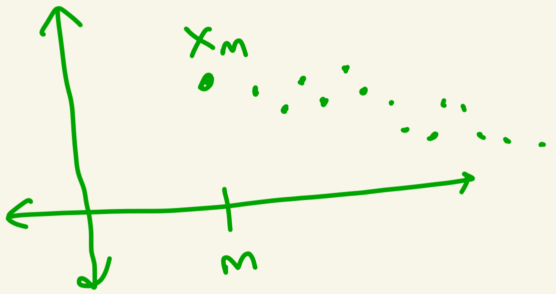
subsequence: $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \ldots$

Monotone subsequence theorem:

If (a_n) is a sequence of real numbers, then there is a subsequence of (a_n) that is monotone.

proof:

We say that the m-th term a_m is a "peak" of our sequence if $a_m \geqslant a_n$ for all $n \geqslant m$.



Case 1: Suppose (a_n) has infinitely many peaks. Then listing the peaks by increasing subscripts we get a subsequence of peaks:

$$x_{m_1} \geqslant x_{m_2} \geqslant x_{m_3} \geqslant x_{m_4} \geqslant \cdots$$

with $m_1 < m_2 < m_3 < m_4 < \cdots$

So there is a monotonically decreasing subsequence.

Case 2: Suppose (a_n) has finitely many peaks.

Set $s_1 = 1$ if there are no peaks.

Otherwise define s_1 as follows.

Let the peaks be listed by increasing subscripts:

$$a_{m_1} \geq a_{m_2} \geq \dots \geq a_{m_r}$$

where x_{m_r} is the last peak.

Set $s_1 = m_r + 1$.

Thus, a_{s_1} is the term immediately after the last peak.

So, a_{s_1} is not a peak.

Thus, there exists a_{s_2} with

$$a_{s_1} < a_{s_2} \quad \text{and} \quad s_1 < s_2.$$

Then again since a_{s_2} is not a peak there exists a_{s_3} with $a_{s_2} < a_{s_3}$ and $s_2 < s_3$.

Continuing in this way we get a subsequence

$$a_{s_1} < a_{s_2} < a_{s_3} < a_{s_4} < \dots$$

with $s_1 < s_2 < s_3 < s_4 < \dots$

Thus, we have a monotone subsequence. $\blacksquare$

Bolzano-Weierstrass:

Let (a_n) be a bounded sequence of real numbers. Then there exists a subsequence that converges.

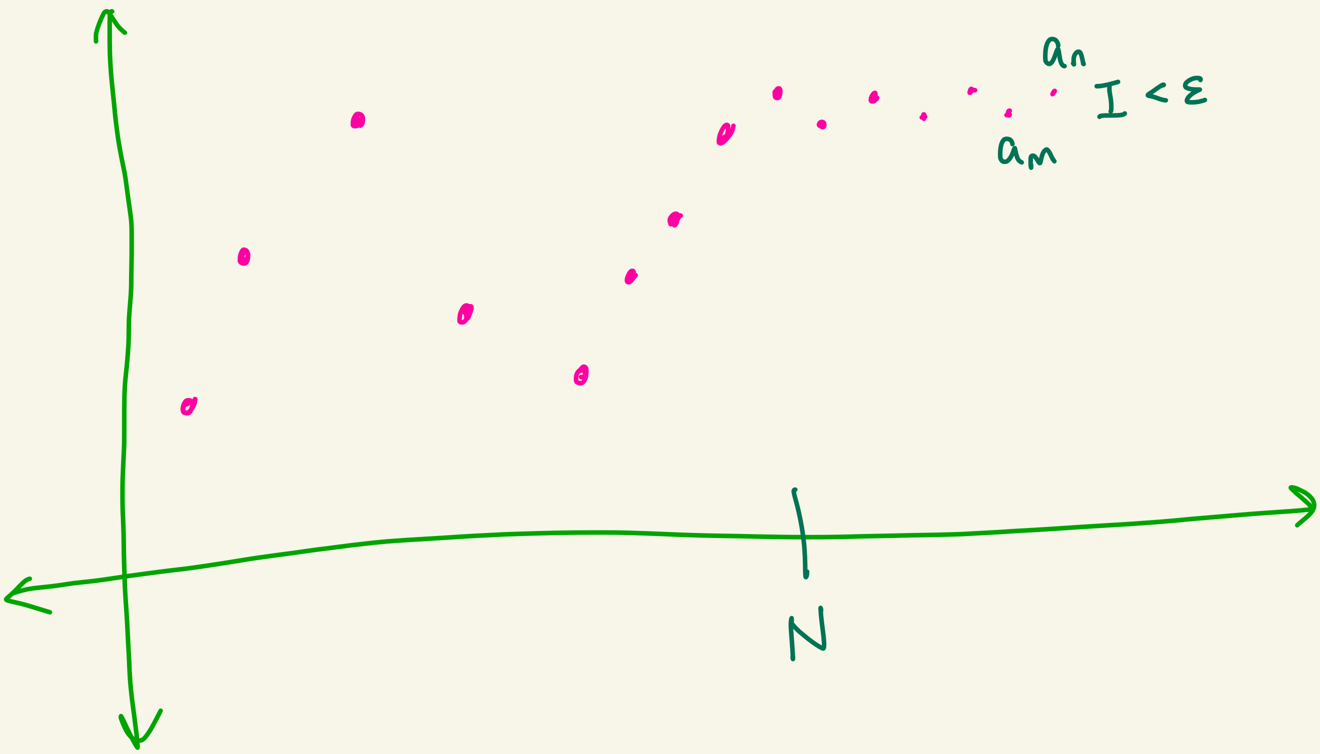
Proof: Let (a_n) be a bounded sequence. By the Monotone subsequence theorem there exists a monotone subsequence (a_{n_k}). Since (a_{n_k}) is a bounded monotone sequence it must converge by the monotone convergence theorem. $\blacksquare$

Ex: $a_n = (-1)^n$ is a bounded sequence.

bounded sequence: $1, -1, 1, -1, 1, -1, 1, -1, \ldots$

convergent subsequence: $1, 1, 1, 1, 1, 1, 1, \ldots$

<u>Def:</u> Let (a_n) be a sequence of real numbers. We say that (a_n) is a <u>Cauchy sequence</u> if for every $\varepsilon > 0$ there exists $N > 0$ where if $n, m \geq N$ then $|a_n - a_m| < \varepsilon$.



Ex: $\left(\frac{1}{n}\right)_{n=1}^{\infty}$ is Cauchy.

proof:

Let $\varepsilon > 0$.

Pick N so that $N > \frac{2}{\varepsilon}$.

This makes $\frac{1}{N} < \frac{\varepsilon}{2}$.

So if $n, m \geq N$ then

$$\begin{aligned}
\left|\frac{1}{n} - \frac{1}{m}\right| &\leq \left|\frac{1}{n}\right| + \left|\frac{1}{m}\right| \\
&= \frac{1}{n} + \frac{1}{m} \\
&\leq \frac{1}{N} + \frac{1}{N} \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&= \varepsilon.
\end{aligned}$$

$\blacksquare$

Theorem: Let (a_n) be a sequence of real numbers. Then, (a_n) converges if and only if (a_n) is Cauchy.

Proof:

($\Rightarrow$) Let (a_n) be a convergent sequence with $\lim_{n\to\infty} a_n = L$.

Let $\varepsilon > 0$.

Then, there exists $N > 0$ where if $k \geq N$ then $|a_k - L| < \frac{\varepsilon}{2}$.

Thus, if $n, m \geq N$, then

$$\begin{aligned} |a_n - a_m| &= |a_n - L + L - a_m| \\ &\leq |a_n - L| + |L - a_m| \\ &= |a_n - L| + |a_m - L| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon. \end{aligned}$$

So, (a_n) is Cauchy.

($\Leftarrow$) Now suppose that (a_n) is Cauchy.

By HW, we get that (a_n) is bounded.

By Bolzano-Weierstrass there exists a subsequence (a_{n_k}) that converges to some $L \in \mathbb{R}$.

Let $\varepsilon > 0$.

Since (a_n) is Cauchy, there exists $N > 0$ where if $m, l \geq N$ then $|a_m - a_l| < \frac{\varepsilon}{2}$.

Since (a_{n_k}) converges to L there exists $n_{k_0} \geq N$ where $|a_{n_{k_0}} - L| < \frac{\varepsilon}{2}$.

Thus, if $n \geq N$ we get

$$
\begin{aligned}
|a_n - L| &= |a_n - a_{n_{k_0}} + a_{n_{k_0}} - L| \\
&\leq |a_n - a_{n_{k_0}}| + |a_{n_{k_0}} - L| \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&= \varepsilon.
\end{aligned}
$$

Thus, (a_n) converges to L. ///

Note that the $(\Leftarrow)$ direction above depended on the completeness of $\mathbb{R}$.